Entanglement of Collectively Interacting Harmonic Strings

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We study the ground-state entanglement of one-dimensional harmonic strings that are coupled to each other by a collective interaction as realized e.g. in an anisotropic ion crystal. Due to the collective type of coupling, where each string interacts with every other one in the same way, the total hamiltonian has a vanishing energy gap and thus shows critical behavior even though the isolated harmonic strings are gapped and thus non-critical. We derive lower and upper bounds for the entanglement, quantified by the von Neumann entropy, between a compact block of oscillators and its environment. For sufficiently large size of subsystems the bounds coincide and show that the area law for entanglement is violated by a logarithmic correction.

Presently there is a growing interest in the interrelation between entanglement and quantum properties of the ground state of many-body lattice models. For a number of spin models [1] a strict correspondence between the absence of criticality, the presence of an energy gap, and an area law for the entanglement was established. The latter states that the entanglement of a compact sub-set of lattice sites with the rest of the system, measured by the von Neumann entropy of the subsystem, scales with the surface area of the sub-set. For critical spin systems it was shown that an additional logarithmic correction to the area law emerges. A similar relation between criticality and entanglement was suggested for harmonic lattice models [2, 3]. In [4, 5] an area law was established for harmonic lattice models in arbitrary dimensions with nearest-neighbor coupling which have a gaped spectrum. For finite-range couplings in one dimension a one-to-one correspondence between the validity of the area law and non-criticality was established in [6], and logarithmic corrections were derived for critical systems.

Although the relation between criticality and entropy-area law seems rather universal, there are a number of examples where this relation does not hold [5, 7]. Until now there is no general understanding of the conditions for the validity of an entropy area law in particular in higher dimensions. In the present paper we discuss a specific *gapless* oscillator model with dimension larger than one, for which an exact asymptotic expression for the entropy can be obtained. Since the system is gapless a violation of the area law is expected, and we derive the exact form of the correction term.

Let us consider a set of parallel harmonic chains (see Fig.1) each containing n_x oscillators, with $n_x \to \infty$ in the thermodynamic limit. We will refer to the direction parallel to the chains as x-axis, and to the orthogonal direction as y-axis. The number of parallel chains is denoted as n_y, again with $n_y \to \infty$ in the thermodynamic limit. The oscillators are described by the canonical variables (q_i, p_i), where $i = 1, 2, ...N$ $(N = n_x n_y)$ is a collective index that labels the oscillator. We adopt the following notation: $i = 1, ..., n_x$ correspond to the oscillators in the first chain with growing x coordinate, $i = n_x + 1, ..., 2n_x$ corresponds to oscillators in the second chain and so on.

We consider a quadratic Hamiltonian of the form

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i,j=1}^{N} V_{ij} q_i q_j, \tag{1}$$

with a coupling matrix V. We are interested only in a translationaly invariant coupling, i.e. we assume that the matrix elements of V depend only on the difference of the x coordinates and the difference of the y coordinates. Hence V is a block Toeplitz matrix [8]

$$V = \begin{bmatrix} V_0 & V_1 & V_2 & & V_{n_y} \\ V_1 & V_0 & V_1 & . & \\ V_2 & V_1 & . & . & V_2 \\ & . & . & . & V_1 \\ V_{n_y} & & V_2 & V_1 & V_0 \end{bmatrix}$$

where V_k are again Toeplitz matrices of dimension $n_x \times n_x$. V_0 describes the oscillator coupling within a single chain, V_1 the coupling between neighboring chains, V_2 between next nearest chains etc. For oscillator systems with a quadratic coupling of the form of eq.(1) the ground state $\Psi_0(\mathbf{q}) \sim \exp\left(-\frac{1}{2}\langle \mathbf{q}| V^{1/2} |\mathbf{q}\rangle\right)$ and all its properties, as e.g. the correlation length in position or momentum space, are determined by the square root of V, where $\mathbf{q} = (q_1, q_2, \ldots, q_N)$ is the vector of position variables. The ground state can easily be determined if V is the square of another matrix, which we assume to be again a Toeplitz matrix,

$$V = Z^2/n_y. \tag{2}$$

The normalization factor $1/n_y$ is choosen such that the matrix element of V to remain finite in the thermodynamic limit $N \to \infty$. Assuming Z to be a Toeplitz matrix guarantees that the coupling V is a Toeplitz matrix as well. In order to have a gapless system we furthermore consider Z to be of the form

$$Z = \begin{bmatrix} \Lambda & Q & Q & . & . & Q \\ Q & \Lambda & Q & . & . & Q \\ Q & Q & \Lambda & . & & \\ & . & & & . & Q \\ & & . & & & Q \\ Q & & & Q & Q & \Lambda \end{bmatrix}. \tag{3}$$

Λ and Q are both $n_x \times n_x$ Toeplitz matrices and assumed to be of finite range, i.e. their matrix elements Λ_k and Q_k, where $\Lambda_k \equiv \Lambda_{k=|i-j|} = \langle i|\Lambda|j\rangle$, vanish exactly for $k \geq R$. The finite range of Λ and Q ensures that the coupling V is of finite range within the chains, i.e. in x-direction. We assume furthermore that Λ, Q and $\Lambda - Q$ are positive definite matrices. A simple calculation shows that the ground state of V is degenerate and in the thermodynamic limit $n_x, n_y \to \infty$ has only one nonzero eigenvalue. This means that the total Hamiltonian, Eq. (1), is gapless, and one expects that the entropy area law is broken [6].

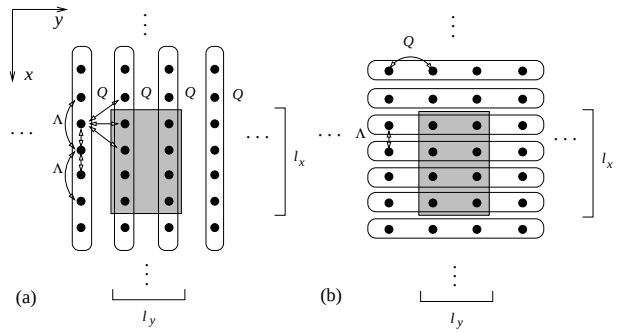

FIG. 1: (a) Collectively interacting strings of harmonic oscillators with finite-range intra-chain coupling Λ and collective inter-chain coupling Q. The grey area indicates the sub-system I of oscillators. (b) Alternative view: interacting strings with collective intra-chain coupling Q and finite-range inter-chain coupling Λ.

Using the spectral representation of V, the correlation matrices $V^{1/2}$ and $V^{-1/2}$ can be decomposed as

$$V^{1/2} = \left[(\Lambda - Q) \otimes \mathbf{1}_y + n_y Q \otimes \mathcal{P}_{n_y,n_y}\right] / \sqrt{n_y}, \qquad (4)$$

and

$$V^{-1/2} = \Big\{(\Lambda - Q)^{-1} \otimes \mathbf{1}_y + \qquad (5)$$
$$+ \left[(\Lambda - Q + n_y Q)^{-1} - (\Lambda - Q)^{-1}\right] \otimes \mathcal{P}_{n_y,n_y}\Big\} \sqrt{n_y},$$

where $\mathbf{1}_y$ is the unity matrix of size $n_y \times n_y$ and $\mathcal{P}_{nm} = |P_{nm}\rangle\langle P_{nm}|$ is the projector onto the (in general non-normalized) vector

$$|P_{nm}\rangle = \frac{1}{\sqrt{n}} \Big(\underbrace{1, 1, ...1}_{m}\Big)^T.$$

Let us now consider a partition of the set of N oscillators into a compact sub-system I with $N_0 = l_x l_y$ and a sub-system II with $N - N_0$ oscillators (see Fig.1). Following Refs. [2, 3, 4, 9], the von-Neumann entropy or the entropy of entanglement of the two parts can be calculated from a decomposition of $V^{1/2}$ into the two subsystems. To this end we express $V^{1/2}$ and $V^{-1/2}$ in a block form according to the two sub-systems by proper reordering of rows and columns

$$V^{-1/2} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}, \qquad V^{1/2} = \begin{bmatrix} D & E \\ E^T & F \end{bmatrix}. \qquad (6)$$

Here A and D are $N_0 \times N_0$ matrices describing the coupling within sub-system I, C and F are $(N - N_0) \times (N - N_0)$ matrices describing the coupling within sub-system II, and the matrices B and E describe the couplings between them. The entropy of entanglement is then given by the eigenvalues $\mu_i \geq 1$ of the matrix product $A \cdot D$ [4]:

$$S = \sum_{i=1}^{N_0} F\left(\sqrt{\mu_i}\right), \qquad (7)$$

$$F(x) = \frac{x+1}{2} \ln \frac{x+1}{2} - \frac{x-1}{2} \ln \frac{x-1}{2}. \qquad (8)$$

If $\mu_i = 1$, the corresponding term in the sum is taken to be zero. Despite the simplicity of its form, expression (7) cannot be explicitly evaluated in general. Due to the special structure of the interaction matrix considered here, the eigenvalues can however be evaluated in the thermodynamic limit.

From the spectral decomposition of $V^{1/2}$, eq.(4), one easily finds that the subsystem matrices read

$$A = \left[A_0 \otimes \mathbf{1}_{l_y} + (A_1 - A_0) \otimes \mathcal{P}_{n_y,l_y}\right] \sqrt{n_y}, \qquad (9)$$
$$D = \left[D_0 \otimes \mathbf{1}_{l_y} + n_y D_1 \otimes \mathcal{P}_{n_y,l_y}\right] / \sqrt{n_y},$$

where A_0, A_1 and D_0, D_1 are $l_x \times l_x$ principal submatrices of $(\Lambda - Q)^{-1}, (\Lambda - Q + n_y Q)^{-1}$, and $(\Lambda - Q), Q$ respectively. For large n_y one has

$$A \cdot D \approx (A_0 \cdot D_0) \otimes \mathbf{1}_{l_y} + n_y (A_0 \cdot D_1) \otimes \mathcal{P}_{n_y,l_y}. \qquad (10)$$

Here we have used that $\mathcal{P}^2_{n_y,l_y} = l_y/n_y \mathcal{P}_{n_y,l_y}$ which scales as $1/n_y$ for fixed l_y and is thus negligible in the thermodynamic limit. Furthermore $\mathcal{P}_{n_y,l_y}$ has one nonzero eigenvalue l_y/n_y, which vanishes in the thermodynamic limit (l_y fixed and $n_y \to \infty$), and $(l_y - 1)$ zero eigenvalues. Thus the $l_x l_y$ eigenvalues of $A \cdot D$ can be decomposed into two sets. The first set consists of the l_x eigenvalues of $A_0 \cdot D_0$ each of which occurs $(l_y - 1)$ times:

$$\begin{aligned} \mu_1, \cdots, \mu_{l_y - 1} &= \alpha_1 (A_0 \cdot D_0), \\ \mu_{l_y}, \cdots, \mu_{2(l_y-1)} &= \alpha_2 (A_0 \cdot D_0), \qquad (11) \\ &\vdots \\ \mu_{(l_x-1)(l_y-1)+1}, \cdots, \mu_{l_x(l_y-1)} &= \alpha_{l_x} (A_0 \cdot D_0). \end{aligned}$$

Here and in the following $\alpha_k(X)$ denotes the kth eigenvalues of the matrix X. The total number of these eigenvalues is $l_x(l_y - 1)$. The second set consists of the l_x eigenvalues of $(A_0 \cdot D_0 + l_y (A_0 \cdot D_1))$

$$\mu_k = \alpha_k (A_0 \cdot D_0 + l_y (A_0 \cdot D_1)), \qquad (12)$$
$$\text{for} \quad k = l_x(l_y - 1) + 1, ..., l_x l_y.$$

Expression (12) for the second set of eigenvalues can be simplified using Lidskii's theorem [10] which states: Let X and Y be M -dimensional Hermitian matrices. Moreover let $\alpha_k(X), \alpha_k(Y)$ and $\alpha_k(X-Y)$, $k=1,...,M$ be the eigenvalues of X, Y and $X-Y$ respectively in ascending order $\{\alpha_1(X) \leq \alpha_2(X) \leq ... \leq \alpha_M(X)\}$. Then there exist numbers $w_{kj} \geq 0$, $(k, j = 1, ..., M)$, such that $\sum_k w_{kj} = \sum_j w_{kj} = 1$ and

$$\alpha_k(X) = \alpha_k(Y) + \sum_{j=1}^{M} w_{kj} \alpha_j(X-Y). \tag{13}$$

Equation (13) implies that for sufficiently large l_y the eigenvalues of the matrix $A_0 \cdot D_0 + l_y (A_0 \cdot D_1)$ are

$$\alpha_k(A_0 \cdot D_0 + l_y (A_0 \cdot D_1)) \approx l_y \sum_{j=1}^{l_x} w_{kj} \alpha_j (A_0 \cdot D_1). \tag{14}$$

An *upper* bound to the entropy can be found by evaluating the sum over the eigenvalues (11) and (12) in eq.(7) separately

$$\begin{aligned} S &= S_1 + S_2 \\ &= \sum_{j=1}^{l_x(l_y-1)} F\left(\sqrt{\mu_j}\right) + \sum_{j=l_x(l_y-1)+1}^{l_x l_y} F\left(\sqrt{\mu_j}\right). \end{aligned} \tag{15}$$

Taking into account eq.(11) one recognizes that S_1 is apart from a prefactor $(l_y - 1)$ formally equivalent to the von-Neumann entropy of a linear oscillator chain of length l_x with interaction $\tilde{V} = (\Lambda - Q)^2$

$$S_1 = (l_y - 1) \sum_{k=1}^{l_x} F\left(\sqrt{\alpha_k(A_0 \cdot D_0)}\right). \tag{16}$$

Since $\Lambda - Q$ was assumed to be strictly positive, the interaction $\tilde{V}$ has only nonzero eigenvalues and thus corresponds to a gaped oscillator chain. As shown in in [4],[6] the entropy of such a linear chain saturates in the thermodynamic limit, i.e it becomes independent on the length l_x of the chain. Thus we have in the thermodynamic limit

$$S_1 \leq l_y c_1. \tag{17}$$

To obtain an upper bound to S_2 we use the inequality $F(x) < 1 - \ln 2 + \ln x$. This yields with eq.(14)

$$S_2 < l_x(1 - \ln 2) + \frac{1}{2} \sum_{k=1}^{l_x} \ln \left(l_y \sum_{j=1}^{l_x} w_{kj} \alpha_j (A_0 \cdot D_1) \right). \tag{18}$$

To further evaluate the last term we make use of the convexity of the logarithm together with the arithmetic mean inequality

$$\begin{aligned} &\frac{1}{2} \sum_{k=1}^{l_x} \ln \left(l_y \sum_{j=1}^{l_x} w_{kj} \alpha_j (A_0 \cdot D_1) \right) \\ &\leq \frac{l_x}{2} \ln \left(\frac{l_y}{l_x} \sum_{j=1}^{l_x} \sum_{k=1}^{l_x} w_{kj} \alpha_j (A_0 \cdot D_1) \right) \\ &. \quad = \frac{l_x}{2} \ln \left(\frac{l_y}{l_x} \sum_{j=1}^{l_x} \alpha_j (A_0 \cdot D_1) \right), \end{aligned} \tag{19}$$

where we have used $\sum_k w_{kj} = 1$ in the last step.

We now have to evaluate the remaining logarithm. For this we make use of the fact that Λ and Q are regular (i.e. strictly positive) Toeplitz matrices. Because of this, their elements can be obtained from the non-negative spectral functions $\lambda(\theta)$ and $q(\theta)$ [11]

$$\Lambda_k = \frac{1}{2\pi} \int_0^{2\pi} \lambda(\theta) \exp[-ik\theta] \, d\theta, \tag{20}$$

$$Q_k = \frac{1}{2\pi} \int_0^{2\pi} q(\theta) \exp[-ik\theta] \, d\theta. \tag{21}$$

Since we have assumed above that also $\Lambda - Q$ is strictly positive, the functions $\lambda(\theta)$, $q(\theta)$ are strictly positive and $\lambda(\theta) > q(\theta)$. In addition, we require also that $(\lambda(\theta) - q(\theta))^{\pm 1}$ and $q(\theta)$ have bounded derivatives of second order. As a consequence one finds (see [11] page 221)

$$\frac{1}{l_x} \left(\sum_{j=1}^{l_x} \alpha_j (A_0 \cdot D_1) \right) \approx \frac{1}{2\pi} \int_0^{2\pi} \frac{q(\theta)}{\lambda(\theta) - q(\theta)} d\theta \tag{22}$$

which is a constant independent on l_x. Thus the desired upper bound to the entropy for sufficiently large l_x, l_y is:

$$S \leq c_1 l_y + c_2 l_x + \frac{l_x}{2} \ln l_y \tag{23}$$

where c_1, c_2 are some constants independent of the size of the subsystem.

A *lower* bound to the entropy can be found from the inequality $F(x) \geq \ln x$. This yields, with the expression (14),

$$\begin{aligned} S \geq\ & \frac{(l_y - 1)}{2} \sum_{k=1}^{l_x} \ln \left[\alpha_k (A_0 \cdot D_0) \right] \\ & + \frac{l_x}{2} \ln(l_y) + \frac{1}{2} \sum_{k=1}^{l_x} \ln \left(\sum_{j=1}^{l_x} w_{kj} \alpha_j (A_0 \cdot D_1) \right). \end{aligned} \tag{24}$$

Making use of Jensen's inequality for concave functions

$\ln\left(\sum_j t_j\alpha_j\right) \geq \sum_j t_j \ln(\alpha_j)$ and $\sum_k w_{kj} = 1$ we find

$$S \;\geq\; \frac{(l_y-1)}{2}\sum_{k=1}^{l_x} \ln\left(\alpha_k \left(A_0 \cdot D_0\right)\right) \qquad (25)$$
$$+\frac{l_x}{2}\ln(l_y) + \frac{1}{2}\sum_{j=1}^{l_x} \ln\left(\alpha_j \left(A_0 \cdot D_1\right)\right).$$

To evaluate the sums over the logarithms we employ Szegö's theorem [11] for determinants of a Toeplitz matrices T. The theorem states: for sufficiently large l_x

$$\ln\left(\det\left(T\right)\right) \approx q_0 l_x + \sum_{k=1}^{\infty} k\left|q_k\right|^2,$$

for regular spectral function $q\left(\theta\right)$. Here q_k is Fourier coefficients of $\ln q\left(\theta\right)$. Since moreover

$$\sum_j \ln\left(\alpha_j (A_0 \cdot D_1)\right) \;=\; \ln\left(\prod_j \alpha_j (A_0 \cdot D_1)\right)$$
$$=\; \ln\big[\det(A_0)\det(D_1)\big], \qquad (26)$$

we eventually find the lower bound

$$S \geq a_1 l_x + a_2 l_y + \frac{l_x}{2}\ln(l_y). \qquad (27)$$

Here a_1, a_2 are constants independent of the size of the subsystem and we have ignored an unimportant constant term.

By combining the two estimates (23) and (27) one finds

$$c_1\, l_x + c_2\, l_y + \frac{l_x}{2}\ln(l_y) \;\geq\; S \;\geq\; a_1\, l_x + a_2\, l_y + \frac{l_x}{2}\ln(l_y).$$

Since both sides of this inequality have the same functional form, S approaches for large l_x, l_y the asymptotic value

$$S \approx \frac{l_x}{2}\ln(l_y), \qquad l_x, l_y \gg 1. \qquad (28)$$

This is the main result of our paper. It shows that the entropy area law is violated for a set of harmonic chains, which for themselves have a gaped spectrum and are non-critical but become gapless by a collective interaction between the strings. Both upper and lower bound to the entropy attain the same logarithmic correction

The break-down of the area law of entanglement in the form given in eq.(28) is intuitively expected from the following simple argument: Let us follow an alternative view and consider harmonic strings in $y-$ rather than in $x-$ direction. The strings couple to each other with finite-range interaction Λ (see Fig.1 b). We thus have reason to assume that $S \sim l_x S_0$, where S_0 is the entropy of a single chain. Since the coupling within the chain is collective (Q), the chain itself is gapless and its entropy scales as $S_0 \sim \ln l_y$. Thus $S \sim l_x \ln l_y$. Due to the anisotropic oscillator coupling the correction term is not symmetric in the length of the boundaries of the rectangular set of oscillators.

A physical system that can be approximated by the model studied here is an anisotropic ion crystal. In such a system the Coulomb-interaction in the direction of the small lattice constant can in first approximation be considered as collective, while the one in an orthogonal direction is of finite range.

In conclusion, we derived an asymptotic expression for the entanglement entropy of a *gapless* system of interacting oscillators in more than one dimension. We found that similar to one-dimensional systems [6] the entanglement area law is violated by a logarithmic correction proportional to the surface area in the critical direction. To our knowledge the system of collectively interacting harmonic strings considered here, which is approximately realized e.g. in an anisotropic ion crystal, is the first non-trivial example of a gapless two-dimensional system for which the correction to the area law can explicitly be calculated.

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